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Tensor Properties and Rotational Symmetry of Crystals. III. Use of Symmetrized Components in Group $3(3_z)$*†

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(*Received* 18 *February* 1981; *accepted* 20 *October* 1982)

Abstract

'Symmetrized' components are introduced in place of the standard ones to improve the method presented in paper I [Fumi & Ripamonti (1980). *Acta Cryst.* A**36**, 535–551]. These components, which are simply related to the standard ones, allow a further reduction of the computational task and also a further simplification of the results and of their use. This is illustrated by application to general two-dimensional tensors of ranks 6 and 8 and by particularization of the results to the cases of the third- and fourth-order elastic tensors.

Introduction

In this paper we introduce symmetrizations in tensor space with respect to the standard reference directions x and y, perpendicular to the principal symmetry axis along the z direction, to improve the method presented in paper I (Fumi & Ripamonti, 1980*a*). From I such symmetrizations allow a further splitting of a tensor invariant in group $3(3_z)$ into independent subtensors: this splitting is additional to the standard ones (see I, § 3*b*) already exploited by the method and concerns only subtensors of even rank in x and y.

* Supported in part by a NATO Research Grant. Part of the 'Tesi di Perfezionamento in Fisica' to be submitted by C. Ripamonti to the University of Genoa.

† We refer the reader to Paper I (Fumi & Ripamonti, 1980*a*) for the details of the method and for the pertinent notations.

1. Symmetrized components

(*a*) *Definition and splitting*

For subtensors of even rank in x and y, we introduce symmetrizations with respect to x,y exchange by defining 'symmetrized components' as follows:

$$c^+ = c + \tilde{c} \tag{1}$$

and

$$c^- = c - \tilde{c} \tag{2}$$

for every pair of standard components c and $\tilde{c}$ related by an x,y exchange. From the identity – except for sign – of the coefficients of c and $\tilde{c}$ in the tensor invariants for group $3(3_z)$ in Hermann's base [see I, § 3*c* and Appendix (iii)], it follows that the c^+'s and c^-'s have non-zero coefficients only in disjoint sets of invariants as follows:

c^+'s of even rank in x and in y (even parity in x and in y),	Re-type invariants of the $n_+ = n_-$ mod 4 subtype	(3)
c^-'s of even rank in x and in y,	Re-type invariants of the $n_+ \neq n_-$ mod 4 subtype	(4)
c^+'s of odd rank in x and in y (odd parity in x and in y),	Im-type invariants of the $n_+ \neq n_-$ mod 4 subtype	(5)
c^-'s of odd rank in x and in y,	Im-type invariants of the $n_+ = n_-$ mod 4 subtype.	(6)

0567-7394/83/020245-07$01.50

In terms of the numerical vector representation of the components, provided by their coefficients in a complete set of invariants, this means that the c^+'s and c^-'s belong to independent subtensors.

By symmetrization with respect to x,y exchange, we have thus split each subtensor of even rank in x and y with given parity in these indices into two 'symmetrized' subtensors identified – according to their parity and +,– symmetrization – by the sets of invariants given in (3) to (6).

(*b*) *Advantages*

Owing to this additional splitting, the symmetrized components further simplify the application of the method given in I and the form of the ensuing results for general tensors.

This splitting greatly reduces the computational task involved in the method by reducing the dimensions of the numerical vector representations used to obtain the resolvent vector equations. Thus at rank 6 one needs to solve only four equations, two of which are coupled, instead of nine equations, of which a set of four, a set of three and a set of two are coupled; and at rank 8 one needs to solve only nine equations, of which two sets of three and one set of two are coupled, instead of 21 equations, of which a set of seven, a set of six, a set of five and a set of three are coupled. One should emphasize that the labour time of solution of these systems of coupled equations increases with the cube of the number of equations.

Reduced dimensions of the vector representations imply also a formal simplification of the results, because of the corresponding reductions in the number of terms in the final expressions.

(*c*) *Uses*

Owing to their simple connection to standard components, symmetrized components can be used almost as standard ones, both in the procedural application of the method given in I and in the practical application of the results.

For a given symmetrized subtensor, a symmetrized component $c^\pm$ is equivalent to the standard component c used to denote it. This is so because of their identical representation in terms of their coefficients in the appropriate set of invariants specified in (3) to (6). Thus, since the method given in I applies separately to each symmetrized subtensor, by using c in place of $c^\pm$, one can still use the pragmatic rules given in I. The only exception is the rule (see I, § 3*g*, second part) used to halve the number of applications of the method by exploiting a correspondence between subtensors of even and odd parity in x and in y, since this rule involves two symmetrized subtensors. Here the actual two-term nature of a symmetrized component implies an additional exchange of +,– symmetrization, owing to the term ending in y.

Symmetrized components are almost as suitable for practical uses as standard components: owing to their simple relationships, they can easily be transformed to standard ones, or they may even be used directly. This is not the case for other non-standard components (see *e.g.* Hermann, 1934; Jahn, 1937; Sirotin, 1961), which are defined through generally complex multiterm combinations, depending on the rank, and thus require in practical use laborious retransformation to the standard ones.

2. Methodological summary

In view of the application in § 3, we summarize the main features of the method given in I, *with special reference to symmetrized components*.*

We recall that the method applies separately to each symmetrized subtensor, and exploits the existence of sets of permutationally connected components (see I, §§ 3*d* and 3*g*, first part) to which we now refer as (permutational) equivalence classes of components. The steps are as follows:

(I) Identification of the equivalence classes of independent components – by correspondence with the equivalence classes of tensor invariants.

(II) For each equivalence class of dependent components, expansion of an arbitrary component into symmetrized combinations of independent ones. After identification of the pertinent symmetrized combinations, this step is accomplished by determination of the expansion coefficients: to this end, one uses a numerical representation of the components in the expansion, which transforms the formal expansion into a numerical vector equation, providing a set of equations for the expansion coefficients.

The pertinent instructions are the following:

(i) The tensor invariants in step I are the invariants in Hermann's base given by

$$n_+ = n_- \bmod 3 \tag{7}$$

(see I, § 3*c*, equation 2) and by the selection rule for the pertinent symmetrized subtensor [(3) to (6)];

(ii) The correspondence in step I is the index correspondence given by

$$+ \leftrightarrow x, \; - \leftrightarrow y \tag{8}$$

over all indices but the last (left free to fix the pertinent parity);

(iii) Symmetrized combinations in step II are obtained by summing over each subset of independent components which are related through a symmetry permutation of the component to be expanded;

* The reader will notice some differences in the rules given here and in I. The rules given here are convenient for $c^\pm$ components, while the rules given in I are convenient for the standard components.

(iv) Numerical representation in step II is the contravariant correspondence (see I, § 3c) given by:

$$c_{\alpha}^{\pm} \leftrightarrow [k_{\alpha 1}, \ldots, k_{\alpha\beta}, \ldots, k_{\alpha n}], \tag{9}$$

where c_{α} is a component in the expansion, $k_{\alpha\beta}$ is the coefficient of c_{α} in the invariant i_{β}, i_{β} is an invariant in (i) corresponding by (ii) to any (independent) component in a symmetrized combination in (iii), n is the number of pertinent symmetrized combinations, and $k_{\alpha\beta}$ is given by

$$k_{\alpha\beta} = (i)^{n_y}(-1)^{n_c}, \tag{10}$$

where n_y is the number of y indices in c_{α}, n_c is the number of $y \leftrightarrow -$ index correspondences between c_{α} and i_{β}.

We recall that expansions of dependent components in the same equivalence class follow from the expansion in step (II) by pertinent permutations (see I, § 3g, first part). We also recall that the actual application of the method is limited to the subtensors of even parity, owing to the correspondence between symmetrized subtensors of different parity:

$$\begin{aligned} (c'\,x)_e^{\pm} &\leftrightarrow (c'\,y)_o^{\mp} \\ (c'\,y)_e^{\pm} &\leftrightarrow -(c'\,x)_o^{\mp} \end{aligned} \tag{11}$$

where the parity (e,o) and the symmetrization (+,—) are explicitly shown, and c' stands for all indices but the last.

To exploit the correspondence (11) we systematically limit permutational equivalence to all indices but the last. [Owing to (11) permutational equivalence cannot extend over all indices in both the corresponding symmetrized subtensors.]

3. Application to general tensors of ranks 6 and 8 in two dimensions

We illustrate the effectiveness of symmetrized components in simplifying both the computational task and the form of the results, by working out the general tensors of ranks 6 and 8 in x and y (see I, § 3h). We give a detailed application only for the c^{+} and c^{-} subtensors of even parity of rank 6, as this example is sufficiently general to illustrate the main procedural points. For the symmetrized subtensors of rank 8 we give only the essential steps schematically.

(A) *Tensor of rank* 6 *in x,y*

This tensor consists of $2^6 = 64$ components which we split into the four symmetrized subtensors and group into the permutational equivalence classes as follows:

even parity c^{+}: $xxxxxx^{+}$; $(xxxxy)\,y^{+}$, 5; $(xxxyy)\,x^{+}$, 10 (12)

even parity c^{-}: $xxxxxx^{-}$; $(xxxxy)\,y^{-}$, 5; $(xxxyy)\,x^{-}$, 10 (13)

odd parity c^{+} : $xxxxxy^{+}$; $(xxxxy)\,x^{+}$, 5; $(xxxyy)\,y^{+}$, 10 (14)

odd parity c^{-} : $xxxxxy^{-}$; $(xxxxy)\,x^{-}$, 5; $(xxxyy)\,y^{-}$, 10. (15)

Owing to the correspondence (11) we apply the method only to the even-parity subtensors.

Even-parity c^{+} subtensor

(I) Identification of the equivalence classes of independent components.

By (7) and (3) the invariants are:

$$\mathrm{Re}(+++--)-,\ 10 \tag{16}$$

and by the index correspondence (8) the independent components are:

$$(xxxyy)\,x^{+},\ 10 \tag{17}$$

where the last index is fixed according to parity.

(II) Expansion of one dependent component of each equivalence class into symmetrized combinations of independent ones.

(a) $xxxxxx^{+}$. By (iii) there is only one symmetrized combination of independent components because all the permutations are symmetry permutations of $xxxxxx^{+}$:

$$\overset{10}{\bar{x}\bar{x}\bar{x}\bar{y}\bar{y}}x^{+}* \tag{18}$$

and the formal expansion is therefore

$$xxxxxx^{+} = c\,\overset{10}{\bar{x}\bar{x}\bar{x}\bar{y}\bar{y}}x^{+}. \tag{19}$$

By (iv), the numerical representation is

$$xxxxxx^{+} \leftrightarrow [1],\ \overset{10}{\bar{x}\bar{x}\bar{x}\bar{y}\bar{y}}x^{+} \leftrightarrow [6-4] = [2], \tag{20}$$

where we have summed the representatives of the components in $\overset{10}{\bar{x}\bar{x}\bar{x}\bar{y}\bar{y}}x^{+}$, and the representatives have been computed through (10) from the invariant Re $+++---$, corresponding by (8) to the independent component $xxxyyx^{+}$. By (20) the formal expansion (19) transforms into the numerical equation for c:

$$[1] = c[2] \tag{21}$$

yielding $c = \frac{1}{2}$, or

$$xxxxxx^{+} = \tfrac{1}{2}\,\overset{10}{\bar{x}\bar{x}\bar{x}\bar{y}\bar{y}}x^{+}. \tag{22}$$

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* Short bars *over* indices denote symmetrization on these indices, *i.e.* summation over their distinct permutations: the number above the component is the number of terms in the sum (see I, p. 543, footnote ||).

(*b*) $xxxxyy^+$. By (iii) there are two symmetrized combinations since only the permutations of the first four indices are symmetry permutations of $xxxxyy^+$:

$$\overset{4}{\overline{xxxy}}yx^+ \quad \text{and} \quad \overset{6}{\overline{xxyy}}xx^+ \tag{23}$$

and the formal expansion is thus

$$xxxxyy^+ = c_1\,\overset{4}{\overline{xxxy}}yx^+ + c_2\,\overset{6}{\overline{xxyy}}xx^+. \tag{24}$$

By (iv) – with reference to the invariants Re $+++---$ and Re $++--+-$ and using (10) – we find the following numerical representation of the terms in (24):

$$\begin{aligned} &xxxxyy^+ \leftrightarrow [-1, 1] \\ &\overset{4}{\overline{xxxy}}yx^+ \leftrightarrow [3-1, 2-2] = [2, 0] \\ &\overset{6}{\overline{xxyy}}xx^+ \leftrightarrow [3-3, 4-2] = [0, 2], \end{aligned} \tag{25}$$

where we have summed the numerical vectors of all the components in each symmetrized combination. By (25) the formal expansion (24) transforms into the numerical vector equation

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 2 \end{bmatrix} \tag{26}$$

yielding $c_1 = -\frac{1}{2}$, $c_2 = \frac{1}{2}$, or

$$xxxxyy^+ = -\tfrac{1}{2}\overset{4}{\overline{xxxy}}yx^+ + \tfrac{1}{2}\overset{6}{\overline{xxyy}}xx^+. \tag{27}$$

Even-parity c^- subtensor

(I) According to (7) and (4) there is only the invariant Re $++++++$ and by (8) we take $xxxxxx^-$ as independent component.

(II) Since there is only one independent component, obviously there are no symmetrized combinations to be constructed. Furthermore, owing to the (unusual) permutational symmetry of this component on all indices, it is sufficient to work out the expansion of one dependent component of the equivalence class $(xxxxy)y^-$, *e.g.* $xxxxyy^-$, since the expansion of any dependent component of the equivalence class $(xxxyy)x^-$, *e.g.* $xxxyyx^-$, follows by the pertinent permutation.

By (iii) the formal expansion of $xxxxyy^-$ reads simply

$$xxxxyy^- = c\,xxxxxx^-. \tag{28}$$

By (iv) the numerical representation with respect to Re $++++++$ is

$$xxxxyy^- \leftrightarrow [-1],\; xxxxxx^- \leftrightarrow [1]. \tag{29}$$

By (29), (28) becomes

$$[-1] = c[1] \tag{30}$$

yielding $c = -1$, or

$$xxxxyy^- = -xxxxxx^-. \tag{31}$$

By permutation

$$xxxyyx^- = -xxxxxx^-. \tag{32}$$

The work is now completed and we can write down at once – using (22), (27), (31) and (32) – all the expressions of the general tensor of rank 6 as follows:

Even parity in x and in y

$$xxxxxx^+ = \tfrac{1}{2}\overset{10}{\overline{xxxyy}}x^+,\ 1 \tag{33a}$$

$$(xxxxy)\,y^+ = -\tfrac{1}{2}(\overset{4}{\overline{xxxy}}y)\,x^+ + \tfrac{1}{2}(\overset{6}{\overline{xxyy}}x)\,x^+,\ 5 \tag{34a}$$

$$(xxxxy)\,y^- = -xxxxxx^-,\ 5 \tag{35a}$$

$$(xxxyy)\,x^- = -xxxxxx^-,\ 10. \tag{36a}$$

Odd parity in x and in y

$$xxxxxy^- = \tfrac{1}{2}\overset{10}{\overline{xxxyy}}y^-,\ 1 \tag{33b}$$

$$(xxxxy)\,x^- = \tfrac{1}{2}(\overset{4}{\overline{xxxy}}y)\,y^- - \tfrac{1}{2}(\overset{6}{\overline{xxyy}}x)\,y^-,\ 5 \tag{34b}$$

$$(xxxxy)\,x^+ = xxxxxy^+,\ 5 \tag{35b}$$

$$(xxxyy)\,y^+ = -xxxxxy^+,\ 10. \tag{36b}$$

In writing these results, we have used the correspondence (11) to obtain the odd-parity expressions, and the relation among the expressions of dependent components in the same equivalence class.

(*B*) *Tensor of rank* 8 *in x,y*

The tensor consists of $2^8 = 256$ components:

even parity c^+: $xxxxxxxx^+$; $(yyyyyyx)\,x^+$, 7; $(yyyyyxx)\,y^+$, 21; $(yyyyxxx)\,x^+$, 35 (37)

even parity c^-: $xxxxxxxx^-$; $(yyyyyyx)\,x^-$, 7; $(yyyyyxx)\,y^-$, 21; $(yyyyxxx)\,x^-$, 35 (38)

odd parity c^+ : $xxxxxxxy^+$; $(yyyyyyx)\,y^+$, 7; $(yyyyyxx)\,x^+$, 21; $(yyyyxxx)\,y^+$, 35 (39)

odd parity c^- : $xxxxxxxy^-$; $(yyyyyyx)\,y^-$, 7; $(yyyyyxx)\,x^-$, 21; $(yyyyxxx)\,y^-$, 35.(40)

The method is applied only to the two even-parity symmetrized sub-tensors and the correspondence (11) is used to obtain the results for the odd-parity ones.

Even parity c^+ subtensor

(I) Invariants Re(−−−−+++)+, 35.

Independent components $(yyyyxxx)\,x^+$, 35.

(II)

(a) $$xxxxxxxx^+ = c\,\bar y\bar y\bar y\overset{35}{\bar y}\bar x\bar x\bar x x^+$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ [1] & = & c[3] \end{array} \qquad \text{(from Re −−−−++++)}.$$

$$\downarrow \qquad \downarrow$$

$$xxxxxxxx^+ = \tfrac{1}{3}\,\bar y\bar y\bar y\overset{35}{\bar y}\bar x\bar x\bar x x^+$$

(b) $$yyyyyyxx^+ = c_1\,\bar y\bar y\overset{15}{\bar y}\bar y\bar x\bar x xx^+ + c_2\,\bar y\bar y\bar y\overset{20}{\bar x}\bar x\bar x yx^+$$

$$\begin{bmatrix}-1\\1\end{bmatrix} = c_1\begin{bmatrix}-1\\3\end{bmatrix} + c_2\begin{bmatrix}4\\0\end{bmatrix} \left(\text{from}\ \begin{matrix}\text{Re −−−−++++} \\ \text{Re −−−+++−+}\end{matrix}\right).$$

$$yyyyyyxx^+ = \tfrac{1}{3}\,\bar y\bar y\overset{15}{\bar y}\bar y\bar x\bar x xx^+ - \tfrac{1}{3}\,\bar y\bar y\bar y\overset{20}{\bar x}\bar x\bar x yx^+$$

(c) $$yyyyyxxy^+ = c_1\,\bar y\bar y\bar y\overset{5}{\bar y}\bar x xxx^+ + c_2\,\bar x\bar x\bar x\bar y\overset{10}{\bar y}yyx^+ + c_3\,\bar y\bar y\overset{20}{\bar y}\bar x\bar x\underline{x}\underline{y}x^+.^*$$

We use the (unusual) permutational equivalence of the independent components over all six indices, and do not use the numerical representation:

$$\begin{aligned} yyyyyxxy^+ &= P_{68}\,yyyyyyxx^+ \\ &= \tfrac{1}{3}\,\bar y\bar y\overset{15}{\bar y}\bar y\bar x xx\bar x^+ - \tfrac{1}{6}\,\bar y\bar y\overset{20}{\bar y}\bar x\bar x xy\bar x^+ \\ &= \tfrac{1}{3}\,\bar y\bar y\bar y\overset{5}{\bar y}\bar x xxx^+ + \tfrac{1}{3}\,\bar x\bar x\bar x\bar y\overset{10}{\bar y}yyx^+ \\ &\quad -\tfrac{1}{6}\,\bar y\bar y\overset{20}{\bar y}\bar x\bar x\underline{x}\underline{y}x^+, \end{aligned}$$

where P_{68} stands for the exchange permutation of the sixth and eighth indices.

Even-parity c^- subtensor

(I) Invariants Re(−−−−−−+)−, 7;

Re +++++++−.

Independent components $(yyyyyyx)\,x^-$, 7;

$xxxxxxxx^-$.

(II)

(a) $$yyyyyxxy^- = c_1\,xxxxxxxx^- + c_2\,yyyyy\overset{2}{\bar y}\bar x x^- + c_3\,\bar y\bar y\bar y\bar y\bar x yyx^-$$

$$\begin{bmatrix}1\\-1\\1\end{bmatrix} = c_1\begin{bmatrix}1\\1\\1\end{bmatrix} + c_2\begin{bmatrix}-2\\0\\2\end{bmatrix} + c_3\begin{bmatrix}-5\\5\\3\end{bmatrix} \left(\text{from}\ \begin{matrix}\text{Re +++++++−} \\ \text{Re −−−−−−+−} \\ \text{Re −−−−+−−−}\end{matrix}\right).$$

$$yyyyyxxy^- = \tfrac{2}{3}\,xxxxxxxx^- + \tfrac{2}{3}\,yyyyy\overset{2}{\bar y}\bar x x^- - \tfrac{1}{3}\,\bar y\bar y\bar y\overset{5}{\bar y}\bar x yyx^-$$

(b) $$yyyyxxxx^- = c_1\,xxxxxxxx^- + c_2\,yyyy\bar y\overset{3}{\bar y}\bar x x^- + c_3\,\bar y\bar y\overset{4}{\bar x}yyyyx^-$$

$$\begin{bmatrix}1\\1\\-1\end{bmatrix} = c_1\begin{bmatrix}1\\1\\1\end{bmatrix} + c_2\begin{bmatrix}-3\\1\\3\end{bmatrix} + c_3\begin{bmatrix}-4\\4\\2\end{bmatrix} \left(\text{from}\ \begin{matrix}\text{Re +++++++−} \\ \text{Re −−−−−−+−} \\ \text{Re −−−+−−−−}\end{matrix}\right).$$

$$yyyyxxxx^- = \tfrac{1}{3}\,xxxxxxxx^- - \tfrac{2}{3}\,yyyy\underline{y}\overset{3}{\underline{y}}\underline{x}\underline{x}^- + \tfrac{1}{3}\,yyy\underline{x}\overset{4}{\underline{y}}\underline{y}\underline{y}\underline{x}^-$$

The complete expressions for the general tensor of rank 8 are as follows:

Even parity in x and in y

$$xxxxxxxx^+ = \tfrac{1}{3}\,\bar y\bar y\bar y\overset{35}{\bar y}\bar x\bar x\bar x x^+,\ 1 \tag{41a}$$

$$(yyyyyyx)\,x^+ = \tfrac{1}{3}(\bar y\bar y\overset{15}{\bar y}\bar y\bar x\bar x x)\,x^+ - \tfrac{1}{6}(\bar y\bar y\bar y\overset{20}{\bar x}\bar x\bar x y)\,x^+,\ 7 \tag{42a}$$

$$\begin{aligned}(yyyyyxx)\,y^+ &= \tfrac{1}{3}(\bar y\bar y\bar y\overset{5}{\bar y}\bar x xx)\,x^+ + \tfrac{1}{3}(\bar x\bar x\bar x\bar y\overset{10}{\bar y}yy)\,x^+ \\ &\quad -\tfrac{1}{6}(\bar y\bar y\overset{20}{\bar y}\bar x\bar x\underline{x}\underline{y})\,x^+,\ 21\end{aligned} \tag{43a}$$

$$\begin{aligned}(yyyyyxx)\,y^- &= \tfrac{2}{3}\,xxxxxxxx^- + \tfrac{2}{3}(yyyyy\overset{2}{\bar y}\bar x)\,x^- \\ &\quad -\tfrac{1}{3}(\bar y\bar y\bar y\overset{5}{\bar y}\bar x yy)\,x^-,\ 21\end{aligned} \tag{44a}$$

$$\begin{aligned}(yyyyxxx)\,x^- &= \tfrac{1}{3}\,xxxxxxxx^- - \tfrac{2}{3}(yyyy\overset{3}{\bar y}\bar y\bar x)\,x^- \\ &\quad +\tfrac{1}{3}(\bar y\bar y\overset{4}{\bar y}\bar x yyy)\,x^-,\ 35.\end{aligned} \tag{45a}$$

Odd parity in x and in y

$$xxxxxxxy^- = \tfrac{1}{3}\,\bar y\bar y\bar y\overset{35}{\bar y}\bar x\bar x\bar x y^-,\ 1 \tag{41b}$$

$$(yyyyyyx)\,y^- = \tfrac{1}{3}(\bar y\bar y\overset{15}{\bar y}\bar y\bar x\bar x x)\,y^- - \tfrac{1}{6}(\bar y\bar y\bar y\overset{20}{\bar x}\bar x\bar x y)\,y^-,\ 7 \tag{42b}$$

$$\begin{aligned}(yyyyyxx)\,x^- &= -\tfrac{1}{3}(\bar y\bar y\bar y\overset{5}{\bar y}\bar x xx)\,y^- - \tfrac{1}{3}(\bar x\bar x\bar x\bar y\overset{10}{\bar y}yy)\,y^- \\ &\quad +\tfrac{1}{6}(\bar y\bar y\overset{20}{\bar y}\bar x\bar x\underline{x}\underline{y})\,y^-,\ 21\end{aligned} \tag{43b}$$

$$\begin{aligned}(yyyyyxx)\,x^+ &= -\tfrac{2}{3}\,xxxxxxxy^+ - \tfrac{2}{3}(yyyyy\overset{2}{\bar y}\bar x)\,y^+ \\ &\quad +\tfrac{1}{3}(\bar y\bar y\bar y\overset{5}{\bar y}\bar x yy)\,y^+,\ 21\end{aligned} \tag{44b}$$

$$\begin{aligned}(yyyyxxx)\,y^+ &= \tfrac{1}{3}\,xxxxxxxy^+ - \tfrac{2}{3}(yyyy\overset{3}{\bar y}\bar y\bar x)\,y^+ \\ &\quad +\tfrac{1}{3}(\bar y\bar y\overset{4}{\bar y}\bar x yyy)\,y^+,\ 35.\end{aligned} \tag{45b}$$

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* Short bars *under* indices denote symmetrization on these indices (see I, p. 543, upper footnote †).

The final equations (41*a*) to (45*b*) are much simpler than the corresponding equations reported in I, Table 2, p. 549. One still has five equations for even parity in x and in y, and five for odd parity, but the number of terms on the right-hand side now ranges from 1 to 3 rather than from 3 to 7 as before.

4. Particularization to third- and fourth-order elastic tensors in two dimensions

We illustrate the effectiveness of the symmetrized components in particularizing the results for general tensors to physical tensors having additional symmetry with respect to index permutations, by working out the third- and fourth-order elastic tensors in two dimensions.

The process of particularization simply consists in replacing by a single component each of the components which are related through a symmetry index permutation of the physical tensor.

For the elastic tensors we adopt the conventional notation:

$$xx \equiv 1, yy \equiv 2, xy \equiv 6.$$

(*A*) *Third-order elastic tensor in x,y*

This is a physical tensor of rank 6 having additional symmetry with respect to index permutations in the first, second and third pairs of indices, and with respect to permutations of these pairs among themselves.

By particularization of (12) to (15), we obtain the following 12 symmetrized components:

$$\text{even parity } c^+ \colon c^+_{111}, c^+_{112}, c^+_{166} \tag{46}$$

$$\text{even parity } c^- \colon c^-_{111}, c^-_{112}, c^-_{166} \tag{47}$$

$$\text{odd parity } c^+ \colon c^+_{116}, c^+_{126}, c^+_{666} \tag{48}$$

$$\text{odd parity } c^- \colon c^-_{116}, c^-_{126}, c^-_{666}. \tag{49}$$

By particularization of the general expressions (33*a*) to (36*b*) we obtain the following results:

Even parity c^+

From (33*a*)

$$c^+_{111} = \tfrac{1}{2}[2c^+_{112} + 8c^+_{166}] = c^+_{112} + 4c^+_{166}.$$

From (34*a*)

$$c^+_{112} = -\tfrac{1}{2}(4c^+_{166}) + \tfrac{1}{2}(2c^+_{112} + 4c^+_{166}) = c^+_{112}$$

$$c^+_{166} = -\tfrac{1}{2}(c^+_{112} + 3c^+_{166}) + \tfrac{1}{2}(c^+_{112} + 5c^+_{166}) = c^+_{166}.$$

Even parity c^-

From (35*a*)

$$c^-_{112} = -c^-_{111}.$$

From (36*a*)

$$c^-_{166} = -c^-_{111}.$$

Odd parity c^+

From (35*b*)

$$c^+_{116} = c^+_{116}.$$

From (36*b*)

$$c^+_{126} = -c^+_{116}$$

$$c^+_{666} = -c^+_{116}.$$

Odd parity c^-

$$c^-_{126} = c^-_{666} = 0$$

by index permutation symmetry.

Then from (33*b*) (or from 34*b*)

$$c^-_{116} = 0.$$

We have thus the following relations:

$$\left.\begin{aligned} c^+_{111} &= c^+_{112} + 4c^+_{116} \\ c^-_{112} &= c^-_{166} = -c^-_{111} \\ c^+_{126} &= c^+_{666} = -c^+_{116} \\ c^-_{116} &= c^-_{126} = c^-_{666} = 0. \end{aligned}\right\} \tag{50}$$

(*B*) *Fourth-order elastic tensor in x,y*

This is a physical tensor of rank 8 having additional symmetry with respect to index permutations in the first, second, third and fourth pairs of indices, and with respect to permutations of these pairs among themselves.

From (37) to (40) it has 20 components:

$$\text{even parity } c^+ \colon c^+_{1111}, c^+_{1122}, c^+_{1222}, c^+_{1266}, c^+_{2266}, c^+_{6666} \tag{51}$$

$$\text{even parity } c^- \colon c^-_{1111}, c^-_{1122}, c^-_{1222}, c^-_{1266}, c^-_{2266}, c^-_{6666} \tag{52}$$

$$\text{odd parity } c^+ \colon c^+_{1116}, c^+_{2226}, c^+_{1226}, c^+_{2666} \tag{53}$$

$$\text{odd parity } c^- \colon c^-_{1116}, c^-_{2226}, c^-_{1226}, c^-_{2666}. \tag{54}$$

From the general expressions (41*a*) to (45*b*) one obtains the following results:

Even parity c^+

From (41*a*)

$$c^+_{1111} = \tfrac{1}{3}(3c^+_{1122} + 24c^+_{1266} + 8c^+_{6666}).$$

From (42*a*)

$$c^+_{1222} = \tfrac{1}{3}(3c^+_{1122} + 12c^+_{1266}) - \tfrac{1}{6}(12c^+_{1266} + 8c^+_{6666})$$

$$c^+_{1222} = c^+_{1122} + 2c^+_{1266} - \tfrac{4}{3}c^+_{6666}.$$

From (42*a*)

$$c^+_{2266} = \tfrac{1}{3}(c^+_{1122} + 10c^+_{1266} + 4c^+_{6666}) - \tfrac{1}{6}(2c^+_{1122} + 14c^+_{1266} + 4c^+_{6666})$$

$$c^+_{2266} = c^+_{1266} + \tfrac{2}{3}c^+_{6666}.$$

Equation (43*a*) gives the same expressions for c^+_{1222} and c^+_{2266}.

Even parity c^-

$$c^-_{1122} = c^-_{1266} = c^-_{6666} = 0$$

by index permutation symmetry.

From (44*a*)

$$c^-_{1222} = \tfrac{2}{3}c^-_{1111} + \tfrac{2}{3}(2c^-_{2266}) - \tfrac{1}{3}(c^-_{1222} + 4c^-_{2266})$$

$$c^-_{1222} = \tfrac{1}{2}c^-_{1111}.$$

From (44*a*)

$$c^-_{2266} = \tfrac{2}{3}c^-_{1111} + \tfrac{2}{3}(c^-_{1222} + c^-_{2266}) - \tfrac{1}{3}(5c^-_{2266})$$

$$c^-_{2266} = \tfrac{1}{3}c^-_{1111} + \tfrac{1}{3}c^-_{1222} = \tfrac{1}{2}c^-_{1111}.$$

Equation (45*a*) can only give expressions for c^-_{1122}, c^-_{1266}, c^-_{6666} which we know to be zero.

Odd parity c^+

$$c^+_{2226} = c^+_{1116}.$$

From (44*b*)

$$c^+_{1226} = -\tfrac{2}{3}c^+_{1116} - \tfrac{2}{3}(2c^+_{2226}) + \tfrac{1}{3}(5c^+_{2226})$$

$$c^+_{1226} = -\tfrac{1}{3}c^+_{1116}.$$

From (44*b*)

$$c^+_{2666} = -\tfrac{2}{3}c^+_{1116} - \tfrac{2}{3}(2c^+_{2226}) + \tfrac{1}{3}(5c^+_{2226})$$

$$c^+_{2666} = -\tfrac{1}{3}c^+_{1116}.$$

Equation (45*b*) gives the same expressions for c^+_{1226} and c^+_{2666}.

Odd parity c^-

$$c^-_{2226} = -c^-_{1116}.$$

From (41*b*)

$$c^-_{1116} = \tfrac{1}{3}(15c^-_{1226} + 20c^-_{2666}).$$

From (42*b*)

$$c^-_{2226} = \tfrac{1}{3}(3c^-_{1226} + 12c^-_{2666}) - \tfrac{1}{6}(12c^-_{1226} + 8c^-_{2666})$$

$$c^-_{2226} = -c^-_{1226} + \tfrac{8}{3}c^-_{2666}.$$

Thus

$$c^-_{2226} = -c^-_{1116} = 0,$$

and

$$c^-_{1226} = c^-_{2666} = 0.$$

Equation (43*b*) can only give relations between c^-_{1226} and c^-_{2666} which we know to be zero.

We have thus the following relations:

$$\left.\begin{aligned} c^+_{1111} &= c^+_{1122} + 8c^+_{1266} + \tfrac{8}{3}c^+_{6666} \\ c^+_{1222} &= c^+_{1122} + 2c^+_{1266} - \tfrac{4}{3}c^+_{6666} \\ c^+_{2266} &= c^+_{1266} + \tfrac{2}{3}c^+_{6666} \\ c^-_{1122} &= c^-_{1266} = c^-_{6666} = 0 \\ c^-_{1222} &= c^-_{2266} = \tfrac{1}{2}c^-_{1111} \\ c^+_{2226} &= c^+_{1116} \\ c^+_{1226} &= c^+_{2666} = -\tfrac{1}{3}c^+_{1116} \\ c^-_{1116} &= c^-_{2226} = c^-_{1226} = c^-_{2666} = 0. \end{aligned}\right\} \quad (55)$$

The usual relations among the standard components of the elastic tensors considered follow from (50) and (55) by using (1) and (2). In particular, one rederives the relations for the third-order elastic tensor first given by Fumi (1952), and those for the fourth-order elastic tensor reported by Brendel (1979) and by Markenscoff (1979),*† as well as those reported by Chung & Li (1974),† who treated a non-tensorial array for fourth-order elasticity.

* Both Brendel (1979) (who treated the tensor in all the crystallographic groups and the isotropic body) and Markenscoff (1979) (who treated the tensor in all trigonal and hexagonal groups) recurred systematically to an electronic computer. For the ease of treatment of general or physical tensors in groups different from $3(3_z)$ see, however, Fumi & Ripamonti (1980*b*).

† The scheme by Markenscoff (1979) for group $3(3_z)$ unfortunately contains a misprint in the expression of c_{1256}: this should read $c_{1256} = \tfrac{1}{6}(c_{1114} + 3c_{1124})$. The errors in the results of Chung & Li (1974) for group $3(3_z)$ have already been listed in I, footnote to § 5. Brendel (1979) states that Krishnamurty's (1963) invariance equations for the isotropic body are in error: in fact, they are correct for the *non-tensorial array* for which they were written, the same as studied by Chung & Li (1974) (see II, Appendix C). Brendel also quotes Gagnepain & Besson (1975) without mentioning that their results for groups 32, $3m$, $\bar{3}m$ are grossly in error, as they give a wrong number of independent components (23 instead of 28).

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